

Switching Systems Controllability and Control Design

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14. ABSTRACT Motivated by the need of dealing with physical systems that exhibit a more complicated behavior than those normally described by classical continuous and discrete time domains, hybrid systems are getting very popular nowadays. Hybrid systems, i.e., heterogeneous systems that exhibit both continuous and discrete dynamics, are abundant in essentially all areas of engineering and scientific endeavor. Hybrid systems can switch between different operating modes where each mode is governed by its own characteristic dynamical laws. Mode transitions are triggered by variables crossing specific thresholds (state events), by the lapse of certain time periods (time events), or by external inputs (input events). Analysis, synthesis and real time implementation of feedback control algorithms for hybrid systems are key issues facing the control and computer science community. Current research on the topic is focused on new methodologies for control design of complex systems with hybrid nature under various constraints, such as mixture of logic and continuous variables, signal quantization, bandwidth limitation, distributed- sensing and computation, and real time scheduling. Controllability of hybrid systems is a hot topic currently, and despite the numerous papers on the topic efficient numerical algorithms that provide control algorithms is still lacking. In particular, there has been a relevant interest in the analysis and synthesis of so-called switching systems intended as the simplest class of hybrid systems. Unlike general hybrid systems, where the trajectories are allowed to have discontinuous jumps due to some change in either the continuous or the discrete dynamics of the system, the term switching system is used to describe systems in which the change of some operative mode maintains the continuity of the flow of the solution even though not its smoothness. A switching system is composed of a family of different (smooth) dynamic modes such that the switching pattern gives continuous, piecewise smooth trajectories. Moreover, it is assumed that one and only one mode is active at each time instant. We consider two classes of switches: switches-on-time and switches-on-state. The switching-on-time is clearly the simplest of the two and, using the terminology above, it can be considered as an intrinsic (or endogenous) switching scheme in the sense that it involves only changes in the tangent space (switching from one element to another one of the family of vector fields) without need to check what happens on the flow of the solution. In the endogenous switching case one assumes to have complete control over the time of switch, i.e., the time instant in which the switching occurs, and over the switching pattern, i.e., which of the systems is activated(selected) in the switching event. This type of switching is called open-loop switching. The switching-on-state is more complicated: in fact it requires a check on the integral curve of the system in order to decide when to pass from a dynamic mode to another one (exogenous switching). It is an obvious but important observation that any switching-on-state path has a unique corresponding switching-on-time path, however these switching times and switching patterns depend on the state. This type of switching will be called		

closed-loop switching.

Controllability of switching systems has been investigated mostly for the linear case, i.e., when the dynamics in the given modes are linear time invariant (LTI) and the case when arbitrary switching is possible (open-loop switching). By using geometric methods and imbedding linear switching systems in the class of the linear parameter varying systems (LPV) we have obtained controllability results [P1][P5].

In contrast, bimodal systems are special classes of switching systems, where the switch from one mode to the other one depends on the state (closed-loop switching). In the simplest case the switching condition is described by a hypersurface in the state space.

The first result about controllability of bimodal systems was given in [R5], where using tools of the geometric control theory a small time local null controllability condition was given. Bimodal systems with single input and dynamics continuous on the switching surface were considered in [R3]. Although the approaches of the two papers used quite different mathematical tools both of the solutions share the characteristics that they assume unconstrained controllability in directions orthogonal to the switching hypersurface.

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Summary

This report investigates controllability of linear time invariant (LTI) switching systems that are controlled using nonnegative inputs. The report gives some algebraic conditions that guarantees global controllability for this class of systems. It is shown that if the system is globally controllable then the number of necessary switchings to control the system is bounded.

As an application controllability conditions for a class of bimodal linear time invariant (LTI) systems are also given. For a certain class of bimodal systems it is shown that controllability is equivalent with controllability of an open-loop switching system using nonnegative controls, i.e. to the controllability of a constrained open-loop switching system.

The paper consider the (closed-loop) stabilizability problem of controlled linear switched systems. It is shown that if the switching system is completely controllable then it is stabilizable. Moreover, it is shown that for these systems a closed-loop (event driven) switching strategy can be found with suitable linear feedbacks that (weakly) stabilizes the system, i.e. the switching system is stabilizable by a generalized piecewise linear feedback. These results holds for systems where the control inputs are sign constrained, too.

It is shown that the completely controllable sampled switching systems with unconstrained inputs can be robustly stabilized (against disturbances and model uncertainties) with suitable linear state feedbacks and a periodic switching strategy. The switching strategy and the feedback gains – by solving a suitable set of linear matrix inequalities (LMI) – can be computed.

Chapter 1

Introduction

Motivated by the need of dealing with physical systems that exhibit a more complicated behavior than those normally described by classical continuous and discrete time domains, hybrid systems have become very popular nowadays. In particular, there has been a relevant interest in the analysis and synthesis of so-called *switching systems* intended as the simplest class of hybrid systems.

A switching system is composed of a family of different (smooth) dynamic modes such that the switching pattern gives continuous, piecewise smooth trajectories. Moreover, we assume that one and only one mode is active at each time instant.

Controllability of switching systems has been investigated mostly for the case when arbitrary switching is possible (open-loop switching) and the objective is to design a proper switching sequence to ensure controllability or stability of (usually) piecewise linear systems, see [1],[69],[85],[88] or [67] for recurrent neural networks. Usually the input set Ω is assumed to be unconstrained, i.e. $\Omega = \mathbb{R}^m$.

Bimodal systems are special classes of switching systems, where the switch from one mode to the other one depends on the state (closed-loop switching). In the simplest case the switching condition is described by a hypersurface \mathcal{C} in the state space. It turns out that for a certain class of bimodal systems controllability question can be reduced to the problem of controllability of sign constrained open-loop switching systems, i.e., the case when $\Omega = \mathbb{R}_+$, see [13]. The multi-input case is typical in process engineering applications where the inputs cannot be negative due to physical reasons.

One of the most elementary constrained controllability problems is that

of the SISO LTI system, with nonnegative inputs, i.e., $\Omega = \mathbb{R}^+$, see [56] for details. The multi-input LTI case, i.e., a special sign constrained switching problem, was solved in [16] and [39], for further insights see [76],[53], [27]. Constrained controllability results for the linear time varying case with continuous right hand side can be found e.g. in [57].

From practical point of view it is important to know if controllability can be performed using a finite number of switchings. It is known that for the unconstrained case and for the constrained case when the small time controllability property holds or the dynamics is continuous the answer is affirmative, [43],[70],[40], moreover in all these cases there exist a bound for the number of switchings.

This report focuses on the controllability problem of LTI switching systems driven by sign constrained control, i.e., the case when $\Omega = \mathbb{R}_+^m$. After recalling some fundamental results from geometrical control theory it will be proved that if the system is globally controllable then one can always use a finite number of switching, moreover, as in the unconstrained situation, the number of necessary switchings is bounded.

The second part of the report provides a global controllability condition that can be used for input sign constrained systems. In contrast to the unconstrained problem where pure Lie algebraic methods can be used effectively to obtain global controllability conditions, in the input sign constrained problem methods borrowed from the theory of convex processes have been proved to be efficient in obtaining global controllability condition formulated in algebraic terms.

Stability issues of switched systems, especially switched linear systems, have been of increasing interest in the recent decade, see for example [22], [44], [45], [46], [48], [70].

In the study of the stability of switched systems one may consider switched systems under given switching signals or tries to synthesise stabilizing switching signals for a given collection of dynamical systems. Concerning the first class a lot of papers focus on the asymptotic stability analysis for switched homogeneous linear systems under arbitrary switching (strong stability, robust stabilization), and provide necessary and sufficient conditions, see [9], [42], [51].

The requirement of (robust) stability imposes very strict conditions on the dynamics, e.g. all the subsystems must be stable or stabilizable. Even under this condition, one has, in general, further restrictions on the allowable switching frequency (dwell time), determined by the spectrum of the

matrices, [82], [81].

For strongly stabilizable linear controlled switching systems the feedback control always can be chosen as a "patchy", linear variable structure controller, see [9]. The control is defined by a conic partition $\mathbb{R}^n = \bigcup_{k=1}^N \mathcal{C}_k$ of the state space while on each cone \mathcal{C}_k the feedback is given by $u = F_k x$.

In the more general situation, when one has unstable modes, more severe conditions on the switching sequence have to be imposed. In this respect one of the most elusive problems is the switching stabilizability problem, i.e., under what condition is it possible to stabilize a switched system by properly designing autonomous (event driven) switching control laws. For autonomous switchings the vector field changes discontinuously when the state hits certain "boundaries". This problem corresponds to the weak asymptotic stability notion of the associated differential inclusions.

Based on the ideas presented in [52] it was proved that the (weak) asymptotic stabilizability of switched autonomous linear systems by means of an event driven switching strategy can be formulated in terms of a conic partition of the state space, see [47],[49]. This result can be seen as a generalization of the corresponding theorem for strong stability. However, in contrast to the strong stability results, the corresponding Lyapunov function is not always convex, see [11].

Completely controllable linear time invariant (LTI) systems $\dot{x} = Ax + Bu$ are stabilizable and the stabilization can be always done by a static state feedback $u = Kx$. Similar result, with a suitable set of linear state feedbacks, is valid for the case when the inputs are sign constrained, see [60], [80].

This report gives a generalization of these fundamental results for the weak stabilizability of the class of completely controllable linear switching systems, where the control inputs might be sign constrained, i.e. it is shown that a completely controllable linear switching system is closed-loop stabilizable, moreover, the stabilization can be performed by using a generalized piecewise linear feedback.

For the class of sampled unconstrained linear switching systems a practical algorithm based on LMIs is given for the computation of the stabilizing switching strategy and stabilizing feedback gains.

Chapter 2

Methods, Assumptions and Procedures

Consider the class of (open-loop)linear switched systems:

$$\dot{x}(t) = A(\sigma(t))x(t) + B(\sigma(t))u(t) \quad (2.1)$$

where $x \in \mathbb{R}^n$ is the state variable, $u \in \Omega \subset \mathbb{R}^m$ is the input variable and $y \in \mathbb{R}^p$ is the output variable. $\sigma : \mathbb{R}^+ \rightarrow S$ is a measurable switching function mapping the positive real line into $S = \{1, \dots, s\}$, i.e., the matrices $A(\sigma)$, $B(\sigma)$ and $C(\sigma)$ are measurable.

A solution (Carathéodory) of (2.1) on an interval I is an almost everywhere differentiable function $\varphi(t) : I \rightarrow \mathbb{R}^n$ that satisfies (2.1) a.e. on I .

State $x \in \mathbb{R}^n$ is *controllable* at time t_0 , if there exist a time instant $t_f > t_0$, a switching function $\sigma : [t_0, t_f] \rightarrow S$, and a bounded measurable input function $u : [t_0, t_f] \rightarrow \mathbb{R}^m$ such that $x(t_f; t_0, x, u, \sigma) = 0$. The switching system (2.1) will be called *completely controllable* or shortly *controllable* if every state is controllable.

State $x \in \mathbb{R}^n$ is *reachable* at time t_0 , if there exist a time instant $t_f > t_0$, a switching function $\sigma : [t_0, t_f] \rightarrow S$, and a bounded measurable input function $u : [t_0, t_f] \rightarrow \mathbb{R}^m$ such that $x(t_f; t_0, 0, u, \sigma) = x$. The switching system (2.1) will be called *completely reachable* or shortly *reachable* if every state is reachable. We will term as reachability set the set of points reachable from the origin, and as controllability set the set of points from which the origin is reachable.

Following classical lines, (2.1) is said to be *globally controllable* if every point in the state space is reachable from any other point in the state space

by using bounded measurable controls and a suitable switching function.

Remark 1 *For unconstrained LTI systems it is reasonable to require that the columns of B_i to be linearly independent, see [84]. In the constrained case, however, we should require that the column vectors be conically independent.*

As an example consider the system $\dot{x} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} u$ with $u \in \mathbb{R}_+^3$. This system is globally controllable but it cannot be globally controlled by using only two, linearly independent, directions.

There is an infinite number of conically independent vectors if the dimension of the space is greater than two!

Remark 2 *For an LTI system the controllability and reachability sets coincide in the unconstrained case but they differ, in general, for the constrained case, compare e.g. $\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u$ with $u \in \mathbb{R}^2$ and $u \in \mathbb{R}_+^2$. In the later situation one has $\mathcal{C} = \{(x_1, x_2) \mid x_1, x_2 \leq 0\}$ while $\mathcal{R} = \{(x_1, x_2) \mid x_1, x_2 \geq 0\}$.*

Recall, that the reachability sets associated to a particular switching sequence are cones, and not subspaces not even convex cones in general, even in the unconstrained case! As an example consider the switching system defined

by $\{(A_1 = 0, B_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}), (A_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, 0)\}$ and the switching sequence

$\sigma = (1, 2)$. Observe, that in contrast to the general linear case, see [43], the attainability set at time T , i.e., the set of points that can be reached from x_0 in exactly T time units, is not a convex cone, in general!

2.1 Switching systems and vector fields

A control system on a smooth n -dimensional manifold M is a collection \mathcal{F} of smooth vector fields depending on independent parameters $w = [w_1, \dots, w_m] \in \Xi \subset \mathbb{R}^m$ called control inputs such that $w(t)$ belongs to a suitable class of real valued functions \mathcal{W} , called admissible controls.

A switching system can be considered as a nonlinear polysystem of the form

$$\dot{x} = f(x(t), w(t)), \quad x(0) = 0 \quad (2.2)$$

where in general, it is assumed that $x \in M$ and $f(., w)$, $w \in \Xi$ is an analytic (smooth) vector field on M . It is supposed that M is an n -dimensional real analytic manifold (para-compact and connected).

In our case $\Xi = S \times \Omega$ with $f_w(x) = f(x(t), w(t)) = A_i x(t) + B_i u$ where $w = (i, u)$. To system (2.2) can be associated in a natural way the collection of vector fields $V_f = \{f_w \mid w \in \Xi\}$, that can be used e.g. in a Lie algebraic treatment, quite suitable for unconstrained problems and small time local controllability problems.

Associated with the system (2.2), denote by $\mathcal{A}_{\mathcal{F}}(x, t)$ the set of all elements attainable from x at time t . For each $x \in M$, $\mathcal{A}_{\mathcal{F}}(x) = \cup_{t \geq 0} \mathcal{A}_{\mathcal{F}}(x, t)$.

The controlled differential equation (2.2), hence the switching system (2.1), is often seen as a differential inclusion. By the Filippov–Ważewski relaxation theorem the solution set of the differential inclusion defined by (2.1) is dense in the set of relaxed solutions, i.e., the solutions of the differential inclusion whose right hand side is the convex hull of the original set valued map, see [5], [71], [18]. This implies that the corresponding attainable sets coincides. Hence, instead of the controllability problem defined for the original switching system (2.1) one can consider the controllability problem associated to the convexified differential inclusion $\dot{x} \in A_c(x)$, where

$$A_c(x) = \sum_{i=1}^s \alpha_i (A_i x + B_i u)$$

and $\alpha_i \geq 0$ and $\sum_{i=1}^s \alpha_i = 1$.

We would like to decide (global) controllability by just examining the vector fields that define a control system without the necessity of obtaining solutions of any kind of the given system. We have already seen that it is possible to "expand" the available vector fields, e.g. by convexification, without changing the system itself, obtaining equivalent descriptions of the same system.

The following section reveals that introducing more and more redundancy in this description – by enlarging the set of vector fields that describes the system, is very useful in deciding the controllability question.

2.2 Lie saturate

The Lie bracket of two vector fields f and g is denoted by $[f, g]$. Under the Lie bracket, and the pointwise addition, the space of all analytic vector fields

on M becomes a Lie algebra; $Lie(\mathcal{F})$ denotes the subalgebra generated by \mathcal{F} . For each $q \in M$, $Lie_q(\mathcal{F})$ is a subspace of T_qM , the tangent space of M at q . A set of vector fields \mathcal{F} on a connected smooth manifold M is called *bracket-generating* (full-rank) if $Lie_q\mathcal{F} = T_qM$ for all $q \in M$.

Families of vector fields \mathcal{F} and \mathcal{G} are said to be (strongly) *equivalent* if $Lie(\mathcal{F}) = Lie(\mathcal{G})$ and $\overline{\mathcal{A}_{\mathcal{F}}(q, T)} = \overline{\mathcal{A}_{\mathcal{G}}(q, T)}$ for all $q \in M$ and for all $T > 0$, where the overbar denotes the closure of the sets. The Lie Saturate $LS(\mathcal{F})$ of a family of vector fields \mathcal{F} is the union of families strongly equivalent to \mathcal{F} .

In general it is difficult to construct the Lie saturate explicitly, however one can construct a completely ascending family of compatible vector fields – *Lie extension* – starting from a given set \mathcal{F} of vector fields. A vector field f is called compatible with the system \mathcal{F} if $\mathcal{A}_{\mathcal{F} \cup f}(q) \subset \overline{\mathcal{A}_{\mathcal{F}}(q)}$ for all $q \in M$. Since $LS(\mathcal{F})$ is a closed convex positive cone in $Lie(\mathcal{F})$, a possibility to obtain compatible vector fields is extension by convexification, see [36]: for $f_1, f_2 \in \mathcal{F}$ and any nonnegative functions $\alpha_1, \alpha_2 \in C^\infty(M)$ the vector fields $\alpha_1 f_1 + \alpha_2 f_2$ is compatible with \mathcal{F} . If $LS(\mathcal{F})$ contains a vector space \mathcal{V} , then $Lie(\mathcal{V}) \subset LS(\mathcal{F})$.

The importance of the Lie extension for controllability is given by the following result, [3]:

Theorem 1 *If \mathcal{F} is a bracket-generating system such that the positive convex cone generated by \mathcal{F} , i.e., $cone(\mathcal{F}) = \{ \sum_{i=1}^k \alpha_i f_i \mid f_i \in \mathcal{F}, \alpha_i \in C^\infty(M), \alpha_i \geq 0, k \in \mathbb{N} \}$ is symmetric, i.e., $cone(\mathcal{F}) = cone(-\mathcal{F})$, then \mathcal{F} is completely controllable.*

For further details on the role of the Lie saturates on controllability see also Chapter 3, Theorem 12. of [36],[3].

Remark 3 *In view of these facts the set of vector fields $\{A_i x + B_i u \mid u \in \Omega\}$ can be replaced, e.g., for the sign constrained case ($\Omega = \mathbb{R}_+^m$), by the set $\{A_i x + b \mid b \in \{0, \lambda e_j B_i, j = 1, \dots, m\}\}$ where e_j are the canonical unit vectors of \mathbb{R}^m and $\lambda > 0$.*

Equivalently the switching system (2.1) can be imbedded in the class of bilinear systems with sign constrained inputs of the form: $\dot{x}(t) = \sum_{k=0}^{(m+1)s-1} \lambda_k F_k(x)$ with $\lambda_k \geq 0$ where $F_{(m+1)l} = A_{l+1}x$ and $F_{(m+1)l+j} = A_{l+1}x + e_j B_{l+1}$, with $l = 0, \dots, s-1$ and $j = 1, \dots, m$.

This embedding gives new insight in the controllability of unconstrained switching systems, too, for details see e.g. Lemma 1 and Theorem 2 of [35].

In what follows it will be assumed that the systems are single input systems. This can be done, since an LTI system with m inputs can be viewed as a switching system formed by m single input systems (bang–bang property).

2.3 Normal controllability

A *trajectory* of the switching system (2.1) will be defined as follows: let $x(t)$ be an absolutely continuous function. We say that $x(t)$ is a (admissible) trajectory of the system (2.1) on $[t_0, t_f]$ if there exists a finite subdivision $t_0 < t_1 < \dots < t_{N-1} < t_N = t_f$ of the interval $[t_0, t_f]$, such that on each subinterval (t_{k-1}, t_k) there exists an admissible function u_k such that one has $\dot{x} = A_k x + B_k u_k$.

Which function is considered as admissible, depends on the specific application. Usually it is fixed to be the set of piecewise constant functions, but could be the set of sufficiently smooth functions, too. This definition excludes problematic situations, like Zeno behavior.

Let us denote by $F_w^t x_0$ the solution of the equation $\dot{\xi} = f_w(\xi)$, $\xi(0) = x_0$ on the interval $[0, t]$. Then for a given vector field \mathcal{F} one can consider the trajectories (positive orbits) of the vector field, i.e.

$$\Phi_{\omega, \tau}^{q, T}(x_0) := F_{w_q}^{t_q} F_{w_{q-1}}^{t_{q-1}} \dots F_{w_2}^{t_2} F_{w_1}^{t_1} x_0 \quad (2.3)$$

where $\tau = (t_1, t_2, \dots, t_q)$, $t_i \geq 0$ with $T = \sum_{j=1}^q t_j$ and $f_{w_i} \in \mathcal{F}$ corresponding to the sequence of piecewise constant controls $\omega = (w_1, w_2, \dots, w_q) \in \Xi^q$.

For a switched linear system $f_{w_i}(x) = A_{s_i} x + B_{s_i} u_i$, with $w_i = (s_i, u_i)$. It is immediate that the trajectories of a switching system are the (positive) orbits of the control system defined by the associated vector field. We will suppress the switching sequence $\sigma = (s_1, s_2, \dots, s_q)$ from the notation and denote the flow by $\Phi_{\tau}^q x_0$ for fixed $\mu = (u_1, u_2, \dots, u_q)$ and by $\Phi_{\mu}^q x_0$ for fixed τ .

A point $y \in M$ is called *normally reachable* from an $x \in M$ if there exist a flow such that $\Phi_{\bar{\tau}}^q x = y$ and the mapping $\tau \in \mathbb{R}_+^q \rightarrow \Phi_{\tau}^q(x)$, which is defined in an open neighborhood of $\bar{\tau}$, has rank $n = \dim M$ at $\bar{\tau}$.

As a consequence of the surjective mapping theorem, [6] Theorem 41.6, one has that there is a neighborhood V of y such that the points $z \in V$ are normally reachable points from x . Let us denote by $\mathcal{N}(x)$ the set of normally reachable points. It follows that if $\mathcal{N}(x)$ is not empty, then it has a nonempty interior. In the context of switching systems it means that the

set of points reachable from x by using piecewise constant switchings has nonempty interior.

The (switching) system has the *normal accessibility property* if for every x the set $\mathcal{N}(x)$ is not empty. The system is *normally controllable* if y is normally reachable from x for every $x, y \in M$. In the language of the switching systems if the system is normally controllable then every two point can be joined using a finite number of switchings.

In what follows we will need the following fundamental result, see Theorem 4.3 in [72]:

Theorem 2 *Let \mathcal{F} be a system of \mathcal{C}^r vector fields on the \mathcal{C}^{r+1} manifold M , $1 \leq r \leq \infty$. Then the following conditions are equivalent:*

- i. \mathcal{F} is controllable*
- ii. \mathcal{F} is normally controllable*
- iii. M is connected and, for every $x \in M$, x is normally accessible from x .*

Remark 4 *Further details concerning the relation between controllability and normal controllability can be found in [25], too. In [26] it is proved that globally controllable smooth systems are controllable by using piecewise constant controls. The key point here is that for a globally controllable system every point has the normal accessibility property. Actually the interior points of the reachability set are reachable by piecewise constant controls, for details see [73].*

2.4 Global controllability

Having in mind the result of the previous section we can concentrate now on the controllability problem itself. Let us apply Theorem 1 in the unconstrained situation: by constructing the Lie extension of the vector field $\mathcal{F} = \{A_i x + B_i u \mid u \geq 0\}$, one can observe that $B_i u$ is compatible with \mathcal{F} , i.e., $B_i u \in LS(\mathcal{F})$. Indeed, $B_i u \in \overline{co(\mathcal{F})}$, since $B_i u = \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda}(A_i x + \lambda B_i u)$. If there is a vector $v \in LS(\mathcal{F})$ such that $-v \in LS(\mathcal{F})$, then $\pm A_i v \in LS(\mathcal{F})$, too, see [36].

Using these techniques in the unconstrained case a necessary and sufficient condition for controllability can be given, see [69],[77]:

Theorem 3 *The unconstrained switching system is controllable if and only if*

$$\text{rank } \mathcal{R}_{\mathcal{A},\mathcal{B}} = n, \quad (2.4)$$

i.e., the multivariable Kalman rank condition, holds, where the subspace $\mathcal{R}_{\mathcal{A},\mathcal{B}}$ is defined as

$$\mathcal{R}_{(\mathcal{A},\mathcal{B})} := \text{span} \left\{ \prod_{j=1}^J A_{l_j}^{i_j} B_k \mid k = 1, \dots, s \right\} \quad (2.5)$$

where $J \geq 0$, $l_j \in \{0, \dots, s\}$, $i_j \in \{0, \dots, n-1\}$. Moreover, if one considers the finitely generated Lie-algebra $\mathcal{L}(A_0, \dots, A_s)$ which contains A_0, \dots, A_s , and a basis $\hat{A}_1, \dots, \hat{A}_K$ of this algebra, then

$$\mathcal{R}_{\mathcal{A},\mathcal{B}} = \sum_{k=0}^s \sum_{n_1=0}^{n-1} \dots \sum_{n_K=0}^{n-1} \text{Im} (\hat{A}_1^{n_1} \dots \hat{A}_K^{n_K} B_k). \quad (2.6)$$

Note, that $\mathcal{R}_{\mathcal{A},\mathcal{B}}$ is the minimal subspace invariant for all of the A_i s containing $\mathcal{B} = \cup_{i=1}^s \text{Im} B_i$.

For sign constrained input one cannot find easily other compatible vector fields than B_i . Unfortunately, in general it is a hard task to prove complete controllability using Theorem 1. A result that gives a necessary and sufficient condition for the small time controllability, i.e, controllability using arbitrary small time, of the constrained switching system and uses Lie algebraic ideas is [79] and [40]. These results are quite restrictive, since small time controllability requires that the convex cone generated by B_i contain a subspace, i.e., $\overline{\text{co}}(\cup_{i=1}^s B_i) - \overline{\text{co}}(\cup_{i=1}^s B_i) \neq \emptyset$.

These observations motivates the necessity to search for other methods in order to obtain a useful algorithm that might test controllability in the sign constrained case. Since $\mathcal{G} = \overline{\text{co}}\{\mathcal{F}\}$ is the Lie extension of \mathcal{F} the controllability problem for the two vector fields are equivalent.

In the more general setting of the convex processes – the set-valued analogues of linear operators – the input constrained controllability problem for LTI systems was solved in [23]. In the sequel a short overview will be given of these results followed by an extension to the input constrained controllability problem of switching systems.

2.5 Stabilizability and asymptotic controllability

The zero solution of the differential inclusion $\dot{x} \in F(x)$ is called asymptotically [strongly/weakly] stable if [for any solution/there exists a solution] $x(t)$ such that for any $\epsilon > 0$ there is a $\delta > 0$ and $\Delta > 0$ such that if $\|x(0)\| < \delta$ then $\|x(t)\| < \epsilon$ holds for all $t \geq 0$ and if $\|x(0)\| < \Delta$ then $\lim_{t \rightarrow \infty} x(t) = 0$ holds.

Concerning the stabilizability problem a lot of papers focus on the asymptotic stability analysis for switched homogeneous linear systems under arbitrary switching, and provide an algebraic characterization in terms of the system matrices, that can be viewed as the requirement imposed by the stability condition on the spectrum of an LTI operator. The requirement of (robust) stability imposes very strict conditions on the dynamics, e.g. all the subsystems must be stable or stabilizable. Even under this condition, one has, in general, further restrictions on the allowable switching frequency (dwell time), determined by the spectrum of the matrices, see e.g. [52],[45],[82].

The relation of strong stabilizability with controllability is trivial (every mode must be controllable). Strong asymptotic stability is equivalent to the existence of a common convex and homogeneous Lyapunov function for every member system. Since this is a very intensively researched area with an impressive number of results of different kind (Lie algebraic, matrix algebraic, etc.) this report does not insist further on this topic.

In the more general situation, when one has unstable modes, more severe conditions on the switching sequence have to be imposed. In this respect one of the most elusive problems is the switching stabilizability problem, i.e. under what condition is it possible to stabilize a switched system by properly designing autonomous switching control laws. This problem corresponds to the weak asymptotic stability notion of the associated differential inclusions.

System (2.2) is *globally asymptotically controllable* (GAC) provided that for each x_0 there is a bounded measurable control such that for the corresponding trajectory $\lim_{t \rightarrow \infty} x(t) = 0$ and $|x(t)| \leq \theta(|x_0|)$ for all $t \geq 0$ for a nondecreasing function $\theta : [0, \infty) \rightarrow [0, \infty)$ with $\lim_{t \rightarrow \infty} \theta(t) = 0$, i.e. for each initial state, there exists a control such that the corresponding solution is defined and converges to zero with "small overshoot" and also that the input remains bounded for x near zero, for details see [54].

The importance of global asymptotical controllability is that it implies

closed-loop stabilizability, i.e. it implies the existence of a feedback control such that the resulting closed loop-system is stable, see [20] (in terms of the so-called π trajectories) and [2], [54] (Carathéodory trajectories).

2.6 Convex processes

A convex process A from \mathbb{R}^n to itself is a set-valued map satisfying $\lambda A(x) + \mu A(y) \subset A(\lambda x + \mu y)$ for all $\lambda, \mu \geq 0$, or, equivalently, a set-valued map whose graph is a convex cone. A convex process is closed if its graph is closed and that it is strict if its domain is the whole space. With a strict closed convex process A one can associate the Cauchy problem for the differential inclusion: $\dot{x}(t) \in A(x(t))$, $x(0) = 0$, for details see [24] and [5].

If $G \subset \mathbb{R}^n$, let us denote by G^+ its (positive) polar cone defined by

$$G^+ = \{p \in \mathbb{R}^n \mid \langle p, x \rangle \geq 0, \forall x \in G\}. \quad (2.7)$$

The transpose A^* of A is defined as the set-valued map defined by $p \in A^*(q) \Leftrightarrow \forall (x, y) \in \text{Graph}(A), \langle p, x \rangle \leq \langle q, y \rangle$. For $\lambda \in \mathbb{R}$ the eigenvectors v of A^* are the nonzero solutions of the inclusion $\lambda v \in A^*(v)$.

Motivated by the terminology used for linear systems we say that A satisfies the *rank condition* if the subspace spanned by the cone $A^k(0)$ is the whole space for some integer $k \geq 1$.

Theorem 4 ([23]) *The following conditions are equivalent:*

- a) *the differential inclusion $\dot{x}(t) \in A(x(t))$, $x(0) = 0$ is controllable,*
- b) *the differential inclusion is controllable at some time $T > 0$,*
- c) *the rank condition is satisfied and A^* has no eigenvectors,*
- d) *for some $k \geq 1$, one has $A^k(0) = (-A)^k(0) = \mathbb{R}^n$.*

Controllability of a linear control system is equivalent to the controllability of the differential inclusion defined by $\dot{x}(t) \in Ax(t) + U$, $x(0) = 0$, with $U = \overline{\text{co}}(B\Omega)$ is a closed convex cone of controls, where $\overline{\text{co}}(S)$ denotes the closure of the convex hull of the set S , see [5]. The adjoint inclusion is $-\dot{q}(t) \in A^T q(t)$, $q(t) \in U^+$, see [23].

2.7 Bimodal systems

Consider a *bimodal piecewise linear system*, i.e., a division of the state space by a hyperplane \mathcal{C} . The dynamics valid within each region is

$$\dot{x}(t) = \begin{cases} A_1 x(t) + B_1 u(t) & \text{if } x \in \mathcal{C}_-, \\ A_2 x(t) + B_2 u(t) & \text{if } x \in \mathcal{C}_+, \end{cases} \quad (2.8)$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $u(t) \in \mathcal{U} \subset \mathbb{R}^m$ is the input vector. The initial state of the system at t_0 is determined by the initial state $x_0 = x(t_0)$ and the initial mode $s_0 \in \{1, 2\}$ in which the system is found at t_0 . \mathcal{C} denotes the hyperplane $\ker C = \{x \mid Cx = 0\}$ and \mathcal{C}_\pm denote the half spaces $\mathcal{C}_+ = \{x \mid Cx \geq 0\}$ and $\mathcal{C}_- = \{x \mid Cx \leq 0\}$. The state matrices are constant and of compatible dimensions, B_1, B_2 having full column rank. $y_s = Cx$ defines the decision vector.

Remark 5 *One can consider a number of different inputs for each mode. In this paper we chose $m_1 = m_2 = m$ for the sake of simplicity but this does not affect the generality of the results.*

Let us suppose that the *relative degree* corresponding to the output y_s and the i th mode is r_i , i.e., $y_s^{(k)} = CA_i^k x$, $k < r_i$ and $y_s^{(r_i)} = CA_i^{r_i} x + CA_i^{r_i-1} B_i u$ with $CA_i^{r_i-1} B_i \neq 0$, see [33]. It is reasonable to assume that $r_i < n$, otherwise it would follow that y_s fulfil a homogeneous differential equation, defined by the characteristic polynomial of A_i . In this case the i th mode would not be able to leave the points of the hypersurface \mathcal{C} , characterized by $y_s = 0$, i.e., such a system would not be well-posed nor completely controllable.

If $r_i < n$ then the system is *right invertible*. Right invertibility denotes the possibility of imposing any sufficiently smooth output function by a suitable input function, starting at the zero state. It turns out that this property is related to $\mathcal{S}_{i,*}$, i.e., the minimal (C_i, A_i) -invariant subspace containing $\text{Im} B_i$. On the other hand *left invertibility*, i.e., the property that for every admissible y_s corresponds uniquely an input u , is closely related to the subspace \mathcal{V}_i^* , the maximal (A_i, B_i) -invariant subspace contained in \mathcal{C} .

For linear systems the points of \mathcal{V}_i^* are not visible by the output. Only the orthogonal projection of the state on the subspace $\mathcal{V}_i^{*,\perp}$ can be deduced from the output and its derivatives, moreover this is the largest subspace where the orthogonal projection of the state can be recognized solely from

the output. If the state is known, the orthogonal projection of the input can be determined modulo $B_i^{-1,T}\mathcal{V}_i^*$, see [7].

Having a single output, in order to remove the ambiguity in the right inverse, one can always redefine the inputs of the system. Indeed, define an input transformation $M_i u = \begin{bmatrix} \tilde{u}_i \\ w_i \end{bmatrix}$ such that $B_i M_i^{-1} = [\tilde{B}_i \ b_i]$ with $CA_i^{r_i-1}\tilde{B}_i = 0$ and $CA_i^{r_i-1}b_i = 1$, e.g., by considering the basis $\{b_i, \tilde{b}_{i,j} = b_{i,j} - CA_i^{r_i-1}b_{i,j}b_i, j = 2, \dots, m\}$ in $\text{Im } B_i$. Then the single input single output (SISO) subsystem (A_i, b_i, C) is left and right invertible, i.e., $\tilde{\mathcal{V}}_i^* \cap \tilde{\mathcal{S}}_{i,*} = 0$ and $\tilde{\mathcal{V}}_i^* + \tilde{\mathcal{S}}_{i,*} = \mathbb{R}^n$, see [8], where the invariant subspaces correspond to the SISO system, while the remaining subsystem (A_i, \tilde{B}_i, C) is not invertible.

It follows that the i^{th} mode can be transformed, see [12] and the "Four Map Theorem" in [8], to :

$$\begin{bmatrix} \dot{\eta}_i \\ \dot{\xi}_i \end{bmatrix} = \begin{bmatrix} P_i \eta_i + R_i y_s + Q_i \tilde{u}_i \\ A_{r_i} \xi_i + B_{r_i} v_i \end{bmatrix} \quad (2.9)$$

$$y_s = C_{r_i} \xi_i, \quad (2.10)$$

where $\eta_i \in \mathcal{V}_i^*$ and the subsystem for ξ_i is a chain of integrators with $B_{r_i} = [1 \ 0 \ \dots \ 0]^T$ and $C_i = [0 \ \dots \ 0 \ 1]$. The inputs v_i and w_i are related as $v_i = CA_i^r x + w_i$.

Since y_s is common for both systems, if $r_1 = r_2 = r$ then $\xi_1 = \xi_2 = \xi$. Recall that the components of ξ are formed by y_s and its derivatives up to order $r - 1$. It follows that the complementer subspaces (zero dynamics) have the same dimension, i.e., there exist a basis transformation T such that $\eta_2 = T\eta_1 = T\eta$. In this case the bimodal system can be written as

$$\dot{\eta} = \begin{cases} P_1 \eta + R_1 y_s + Q_1 \tilde{u}_1 & \text{if } y_s \geq 0 \\ P_2 \eta + R_2 y_s + Q_2 \tilde{u}_2 & \text{if } y_s \leq 0 \end{cases} \quad (2.11)$$

$$\dot{\xi} = \begin{cases} A_r \xi + B_r v_1 & \text{if } y_s \geq 0 \\ A_r \xi + B_r v_2 & \text{if } y_s \leq 0 \end{cases} \quad (2.12)$$

Remark 6 Observe that the required transformation can be performed by the same change of base in the state space. e.g., $\begin{bmatrix} \eta \\ \xi \end{bmatrix} = Tx$, where for the last rows of T are chosen the vectors CA_2^j , $j = 0, \dots, r-1$. However the feedback to obtain the desired structure might differ. The input transformations are also different, in general; this difference is reflected in the notation u_1, u_2 and v_1, v_2 , respectively.

Since the decomposition – i.e., the transformation T – depends only on C, A and r , the choice of the input transformation does not play any role in the validity of the controllability results.

In the case when $r_1 \neq r_2$ such a splitting is not possible but the system can be transformed into (suppose that $r_1 < r_2$):

$$\dot{\eta} = \begin{cases} P_1\eta + R_1y_s + Q_1\tilde{u}_1 & \text{if } y_s \geq 0 \\ P_2\eta + R_2y_s + Q_2\tilde{u}_2 + Q_3v_2 & \text{if } y_s \leq 0 \end{cases} \quad (2.13)$$

$$\dot{\xi} = \begin{cases} A_r\xi + B_rv_1 & \text{if } y_s \geq 0 \\ A_r\xi + B_r\bar{\eta} & \text{if } y_s \leq 0, \end{cases} \quad (2.14)$$

where $\bar{\eta}$ denotes the last component of η .

In contrast to the previous situation, in this case the subsystem ξ , hence the decision variable y_s , cannot be controlled independently from the subsystem η in both modes. Moreover, in the first mode the only way to control the higher order derivatives of y_s is through the inputs \tilde{u}_1 . This fact makes the study of the controllability problem for these systems, in general, more difficult.

Problem formulation

In this report it is addressed the case when $r_i = r$, for which the system is always well posed, see [31]. For sake of simplicity the results will be presented for the case when $r = 1$, i.e.,

$$\dot{\eta} = \begin{cases} P_1\eta + R_1y_s + Q_1u & \text{if } y_s \geq 0 \\ P_2\eta + R_2y_s + Q_2u & \text{if } y_s \leq 0 \end{cases} \quad (2.15)$$

$$\dot{y}_s = v, \quad (2.16)$$

but the assertions remain valid for the general case.

The controllability question of the bimodal system can be reduced to the question of controllability/reachability of the origin through the closed-loop switchings allowed by the switching surface \mathcal{C} . Due to the fact that the bimodal system is not a linear system, the affirmative answer given on this question is not completely trivial.

Reference [79] deals directly with problems described by (2.15) and (2.16), while [28] assumes only single input left and right-invertible systems whose dynamics are smooth, i.e., continuous along the trajectories. In this case one

has $A_1x+B_1u=A_2x+B_2u$, for all $x\in\mathcal{C}, u\in\mathcal{U}$. It follows that $A_2=A_1-KC$ and $B_1=B_2=B$ for a suitable matrix K , i.e., one has $P_1=P_2=P$ and $Q_1=Q_2=0$ in (2.15).

Chapter 3

Results and Discussions

3.1 Finite number of switchings, sampling

The main result of this section can be formulated as:

Proposition 1 *If the switching system (2.1) is globally controllable than it is also globally controllable by using piecewise constant switching functions, i.e. using only a finite number of switchings.*

Moreover, there exist a bound for the necessary number of switchings, that depends only on the system matrices and Ω . There exist a universal (finite) switching sequence σ such that the time varying system $\dot{x} = A(\sigma)x + B(\sigma)u$ is globally controllable.

Proof: The first part of the assertion follows from the implication (i) \Rightarrow (ii) of Theorem 2.

For the second part recall that the reachability set \mathcal{R}_σ associated to a switching sequence σ is a pointed cone. From (iii) of Theorem 2 follows that the origin is normally accessible from itself, hence there is a neighborhood of the origin (a ball) that is also normally accessible by the same switching sequence. It follows that the pointed cone \mathcal{R}_σ contains a ball around the origin, i.e., $\mathcal{R}_\sigma = \mathbb{R}^n$. Since σ contains a finite number of switchings our assertion is proved.

Remark 7 *For unconstrained switching, [70], and for small time controllability of input constrained switching, [40] the finiteness of the switching sequence follows from the fact that the authors consider only piecewise constant switching functions (trajectories) while the boundedness follows from the derivation*

of the controllability condition. The advantage of Proposition 1 is that one can concentrate on the global controllability problem in general, i.e. admitting measurable controls also, which is a common setting for studying controllability of nonlinear systems.

Remark 8 From Proposition 1 follows that there is a minimum number of switching that ensures (global) controllability. It is an open problem to give a (tight) upper bound for this number, even in the unconstrained case, see [70]. It is neither clear if this minimum can be achieved by using a single switching sequence or not.

In a general nonlinear context these questions are posed in the framework of time optimal control. The obtained results are too restrictive to be applicable for a switching system, for details see [41],[74],[78].

In the definition of normal reachability the control input sequence μ is fixed while the switching times may vary in a certain neighbourhood of τ . It turns out that the rank of the map $\mu \in \Omega^N \rightarrow \Phi_\mu^N(x)$ is also significant and it is closely related to the controllability of the sampled system, in general, for details see [63, 62, 65]. A point $y \in M$ will be called *full rank reachable* from an $x \in M$ if there exist a flow such that $\Phi_{\bar{\mu}}^N x = y$ and the mapping $\mu \in U^N \rightarrow \Phi_\mu^N(x)$, which is defined in an open neighborhood of $\bar{\mu}$, has rank $n = \dim M$ at $\bar{\mu}$.

Proposition 2 For the globally controllable linear switching system (2.1) for arbitrary point pairs (x, y) one has that y is full rank reachable from x . Moreover, every point pair can be joined in a full rank reachable way by using the same sequence (σ and τ fixed).

Proof: For unconstrained linear switching systems the assertion is well known, see e.g. [68] or [70]. The constrained case can be reduced to the unconstrained result and Proposition 1: let us consider a point that is full rank reachable from the origin with positive controls. Since the constrained controllable system is also unconstrained controllable, such a point clearly exists. However, by controllability, the origin can be reached from the point z by using a finite switching sequence. By joining these two finite sequences one has that an open neighbourhood of the origin is full rank reachable from the origin. Since the reachability set \mathcal{R}_σ is a pointed cone that contains a ball it follows that $\mathcal{R}_\sigma = \mathbb{R}^n$.

Example 1 To illustrate the difference between the "time topology" related to $\Phi_\tau^q(x)$ and the "input topology" related to $\Phi_\mu^q(x)$ – see [64] for the terminology – consider the switched system defined by the modes $A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $b_1 = 0$ and $A_2 = 0$, $b_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, respectively. The corresponding flows are $F_1^t(x) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x$ and $F_{2,u}^t(x) = x + tu \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. It follows that for any $u > 0$ and $t > 0$ with the switching sequence $\sigma = (2, 1, 2, 1, 2)$, input sequence $\mu = (u, 0, -2u, 0, u)$ and time sequence $\bar{\tau} = (t, t, t, t, t)$ the flow

$$\begin{aligned} \Phi_\tau^5(0) &= F_{2,u}^{t_5} \circ F_1^{t_4} \circ F_{2,-2u}^{t_3} \circ F_2^{t_2} \circ F_{2,u}^{t_1}(0) = \\ &= t_1 u \begin{bmatrix} t_2 + t_4 \\ 1 \end{bmatrix} - 2t_3 u \begin{bmatrix} t_4 \\ 1 \end{bmatrix} + t_5 u \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

has full rank at $\bar{\tau}$ with $\Phi_\tau^5(0) = 0$, see Figure 3.1.

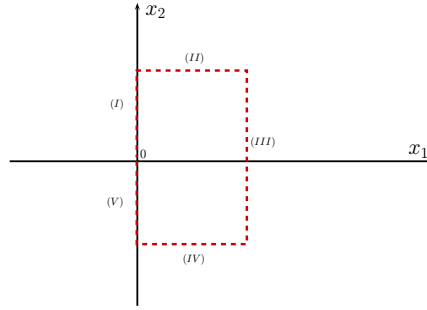


Figure 3.1:

For any $t > 0$ with the switching sequence $\sigma = (2, 1, 2)$, input sequence $\mu = (u_1, 0, u_2)$ and time sequence $\tau = (t, t, t)$ the flow

$$\Phi_\mu^3(x) = F_{2,u_2}^t \circ F_2^t \circ F_{2,u_1}^t(0) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x + tu_1 \begin{bmatrix} t \\ 1 \end{bmatrix} + tu_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

has full rank at any $\bar{\mu}$ with $\Phi_\mu^3(x) = y$ ($\bar{\mu} = (0, 0)$ for $x=y=0$).

Observe that in the input topology, i.e. for the discretized system, the design problem is linear in the unknown variables. This fact motivates that in the investigations of linear switched systems the usage of this topology is preponderant.

From Proposition 2 it is immediate that:

Corollary 1 *For every completely controllable linear switching system (2.1) the sampled discrete-time system is also completely controllable for suitable sampling rates.*

As a consequence one has the following embedding/restriction, see [77] for further details:

Corollary 2 *For every completely controllable linear switching system (2.1) one can associate – not necessary a unique – completely controllable periodic linear time varying system $\dot{x} = A(t)x + B(t)$.*

The non uniqueness comes from the fact that one has more switching sequences σ such that $\mathcal{R}_\sigma = \mathbb{R}^n$. From practical reasons it is preferable to chose the sequence with minimal length. While algorithms that compute suitable σ such that $\mathcal{R}_\sigma = \mathbb{R}^n$ can be relatively easily constructed, obtaining a minimal sequence is an open problem yet.

3.2 Controllability analysis of linear switching systems with sign constrained inputs

Applying Theorem 4 one can obtain the result of Kalman, [37], for the unconstrained case and the results reported in [16] and [39] for the constrained input case.

Along these lines one can find a necessary condition for the controllability of an open-loop switching system with nonnegative control:

$$\dot{x} = A_i x + B_i u, \quad u \geq 0. \quad (3.1)$$

Denote by $U = \overline{\text{co}}(\cup_{i=1}^s B_i \Omega)$ and by

$$A_c(x) = \left\{ \sum_{i=1}^s \alpha_i A_i x + U \mid \alpha_i \geq 0, \sum_{i=1}^s \alpha_i = 1 \right\}.$$

Then the associated differential inclusion have the same reachability set as the original switching system (3.1). As it was shown in the previous section,

this extension is based on the geometrical framework of control theory given by the Lie theoretic approach. Even this system does not define a convex process the result d) of Theorem 4 remains valid for the general case, too.

Let us consider the differential inclusion $\dot{x} \in F(x)$, $x(0) = \xi$ and the reachable set $\mathcal{R}^T(\xi) = \{x(T)\}$ where x is a solution. If F has nonempty, compact, convex values and is locally Lipschitz then

$$\mathcal{R}^T(\xi) = \lim_{N \rightarrow \infty} (I + \frac{T}{N}F)^N(\xi) := [\text{Exp } F](T\xi),$$

for definitions and details see [83].

Extending this result, Proposition 2 of [19] shows that for a positively homogeneous inclusion, $(F(\alpha) = \alpha F(x), \alpha > 0)$, one has

$$[\text{Exp } F](t\xi) = \xi + \sum_{k=1}^{\infty} \frac{t^k}{k!} F^k(\xi),$$

where $F^k = F \circ F \circ \dots \circ F$.

The assertion follows by applying this result for the differential inclusion defined by A_c . Observe that A_c is a positively homogeneous inclusion with closed, convex values, hence $\alpha A_c^k(0) = A_c^k(0)$ for any $\alpha > 0$. Moreover, since

$$A_c^k(x) = \text{co}\{A_i\} A_c^{k-1}(x) + A_c(0)$$

it follows that $A_c^{k-1}(0) \subset A_c^k(x)$ and that $A_c^k(x)$ is a closed convex cone.

It follows that for the reachability set \mathcal{R} one has

$$\mathcal{R} = \cup_{T \geq 0} \mathcal{R}^T(0) = \lim_{N \rightarrow \infty} \sum_{k=1}^N A_c^k(0) = \lim_{N \rightarrow \infty} A_c^N(0).$$

Since the series $A_c^k(0)$ is an increasing sequence of closed convex cones it follows, that if $\mathcal{R} = \mathbb{R}^n$, then there is a finite index M such that $\mathcal{R} = A_c^M(0)$.

Then, the main result of this paper concerning controllability of the input constrained open-loop switching systems can be formulated as:

Proposition 3 *The following conditions are equivalent:*

- a) *the switching system $\dot{x} = A_i x + B_i u$, $i \in \{1, \dots, s\}$, $u \in \Omega$ is controllable,*

- b) the associated differential inclusion $\dot{x} \in A_c(x)$, $x(0) = 0$ is controllable,
c) for some $k \geq 1$, one has $A_c^k(0) = (-A)_c^k(0) = \mathbb{R}^n$.

Since controllability of the constrained system implies controllability of the unconstrained system, the rank condition for $A_c(x)$ can be replaced by the multivariable Kalman rank condition of Theorem 3.

The results of Proposition 3 shows that if controllability conditions are satisfied for a system defined by (A_i, B_i) then they are satisfied for a system defined by $(-A_i, -B_i)$, too. It follows that controllability of an open-loop switching system implies reachability, hence global controllability.

Introducing the notation $\mathbf{co}\{V_j\}$ for the convex hull of the subsets $V_j \subset \mathbb{R}^n$, then the sets $A_p^k := A_c^k(0)$ and $A_m^k := (-A)_c^k(0)$ can be computed using the following algorithm:

Controllability Algorithm:

$$U = \mathbf{co}\{B_i\Omega \mid i = 1, \dots, s\} \quad (3.2)$$

$$A_p^1 = U, \quad A_m^1 = -U, \quad (3.3)$$

$$A_p^{k+1} = \mathbf{co}\{A_i A_p^k + B_i\Omega \mid i = 1, \dots, s\}, \quad (3.4)$$

$$A_m^{k+1} = \mathbf{co}\{-A_i A_m^k - B_i\Omega \mid i = 1, \dots, s\}. \quad (3.5)$$

Example 2 To illustrate the results let us consider the system

$$\begin{aligned} A_1 &= 0, \quad B_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \\ A_2 &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B_2 = 0, \\ A_3 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad B_3 = 0. \end{aligned}$$

Applying the algorithm one can find that $A_p^k = A_m^k$ with $k = 4$, i.e., the system is globally controllable.

Remark 9 *The number k of Proposition 3, i.e., $A_p^k = A_m^k$ is not related with the minimum number of switchings. Unfortunately, for $n \geq 2$ this number can be arbitrarily large, as the following example shows:*

$$A_1 = \begin{bmatrix} 0 & \frac{1}{N} \\ -\frac{1}{N} & 0 \end{bmatrix} \quad B_1 = 0, \quad A_2 = 0, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

By [55] this system is globally controllable by nonnegative controls and $k > N$. However, it is clear that using the switching sequence $\sigma = (2, 1)$ every point is reachable.

Finding an efficient algorithm that can prove or disprove controllability is subject of further research.

3.3 Stabilizability of completely controllable linear switching systems

In order to prove stabilizability of completely controllable linear switching systems it is sufficient to show that they are globally asymptotically controllable.

Lemma 1 *A completely controllable linear switching system is globally asymptotically controllable.*

Proof: Let us consider the unit sphere \mathcal{S} and a point $x \in \mathcal{B}$. By complete controllability it follows that there is a finite switching sequence $\tau_x = (\tau_{L_x}, \dots, \tau_2, \tau_1)$ and a bounded measurable control sequence (actually a piecewise constant control) $u_x = (u_{L_x}, \dots, u_2, u_1) \in \Omega^{L_x}$ such that the corresponding trajectory steers the point x to the origin, i.e.

$$\Phi(\tau_x, u_x)x = \prod_{j=1}^{L_x} e^{(A_{l_j}\xi + B_{l_j}u_j)\tau_j} x = 0,$$

where, for notational convenience $e^{(A_{l_j}\xi + B_{l_j}u_j)\tau_j}\zeta$ denotes the flow associated to the vector field $A_{l_j}\xi + B_{l_j}u_j$ that passes through the initial state ζ at $t = 0$.

By the continuity of the map $\Phi(\tau_x, u_x)$ for the fixed pair (τ_x, u_x) for every $\epsilon > 0$ there is a neighborhood \mathcal{V}_x of x such that

$$\|\Phi(\tau_x, u_x)\xi\| < \epsilon, \quad \forall \xi \in \mathcal{V}_x,$$

hence for all $\xi \in \mathcal{W}_x = \mathcal{V}_x \cap \mathcal{S}$, see Fig. 3.2.

Since the unit sphere is compact, there is a finite covering $\mathcal{S} = \cup_{j \in J} \mathcal{W}_{x_j}$. It follows that there is a control strategy that maps the unit sphere into the sphere with radius $\epsilon < 1$ defined by this finite partition.

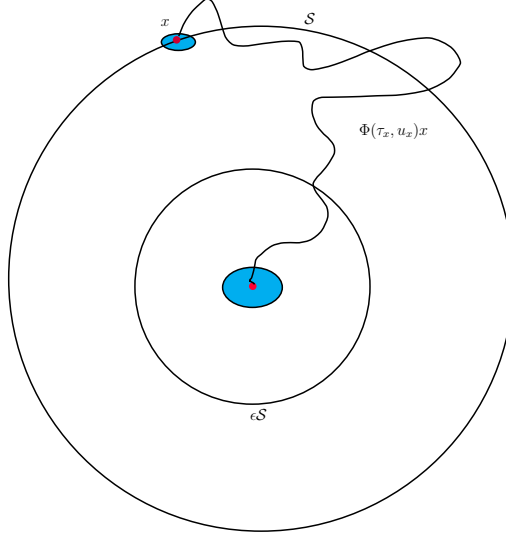


Figure 3.2: Asymptotic controllability

Since the linear maps $\Phi(\tau_{x_j}, u_{x_j})$ are bounded one has a uniform bound for the "overshoot",

$$\Theta = \max_{j \in J} \|\Phi(\tau_{x_j}, u_{x_j})\|.$$

Since the vector fields are linear the reachable spaces are cones, therefore the control strategy can be extended from the unit sphere to the whole state space, i.e. one can construct a trajectory with the bound $\|x(t)\| < \Theta \|x_0\|$ that converges to the origin. It follows that a completely controllable linear switching system is globally asymptotically controllable.

Corollary 3 *The completely controllable linear switching system (2.1) is closed-loop stabilizable.*

Remark 10 *For discrete-time linear switched systems with unconstrained inputs the assertion of Lemma 3 was proved recently, see [86]. The switching strategy in the proposed solution is a periodic one, based on the universal switching sequence. In contrast to the continuous time case the proof is constructive, moreover the necessary linear feedbacks can be obtained by a linear matrix inequality.*

The continuous-time result for the unconstrained input case can be obtained directly from the discrete-time one by using the fact that generically the discretized linear switched system preserves the complete controllability property, see [70]. The resulting control will be a stabilizing control with a periodic (open-loop) switching strategy and a "feedback-like" control for u – a feedback implemented in a sample and hold way.

The assertion of Lemma 3 is also valid for the sign constrained control input case, when the proof based on the discrete-time result is not applicable.

Stabilizability by Generalized Piecewise Linear Feedback

While the general nonlinear theory guarantees the existence of a not too pathological feedback and control Lyapunov function, see [2], [38], [54], the results are hard to be applied to construct directly the required feedback for the switching system, i.e. to obtain the closed-loop switching strategy and necessary control inputs or even to infer that the control inputs are given by linear feedbacks.

Given an autonomous linear switching system

$$\dot{x} = A_i x, \quad i \in S$$

it is a nontrivial task to decide if the system is (weakly) stabilizable or not, in general. There are only a few sufficient conditions that guarantee stabilizability and provide a relatively simple closed-loop switching strategy. One such situation is when the convex hull of the system matrices contains a stable (Hurwitz) matrix, i.e. when there are $\alpha_i > 0$, $\sum_{i=1}^s \alpha_i = 1$ such that $\sum_{i=1}^s \alpha_i A_i$ is stable.

For the nonautonomous case with unconstrained inputs it is known that if the sum of the individual controllability subspaces gives the whole state space, then there are linear state feedbacks $u = K_i x$ such that the resulting

linear switching system

$$\dot{x} = (A_i + B_i K_i)x, \quad i \in S$$

is stable with a suitable closed-loop switching strategy, see [70]. It is not hard to figure out that the required condition is sufficient to guarantee that for any convex combination $\alpha_i > 0$, $\sum_{i=1}^s \alpha_i = 1$ there exist feedbacks K_i such that $\sum_{i=1}^s \alpha_i (A_i + B_i K_i)$ is stable.

As it can be concluded through simple examples, see [70], there are completely controllable switching systems that are not stabilizable by merely applying a single linear state feedback for the individual subsystems. However, as it will be shown in this Section, if the number of linear feedbacks is increased, one can obtain a set of autonomous linear systems that are (weakly) stabilizable.

For a given set of non-autonomous (controlled) linear switched systems (2.1) we call *Generalized Piecewise Linear Feedback Stabilizability* (GPLFS) the problem of finding a closed-loop switching strategy with

- suitable linear feedbacks $u_i = K_i x$, $i \in S$
- a switching law $\kappa(x) \in S$, $x \in \mathbb{R}^n$

that (weakly)stabilizes the system.

The reasoning behind introducing the concept of generalized piecewise linear feedback stabilizability is to separate the task of finding a suitable switching strategy and that of finding suitable control inputs with low complexity that stabilizes the system in closed-loop.

The main idea is to substitute the original stabilizable nonautonomous system by a stabilizable autonomous linear switched system that might contain more modes than the original one, by applying as control inputs a number of suitable static linear control feedbacks.

Theorem 5 *The completely controllable linear switching system (2.1) is generalized piecewise linear feedback stabilizable.*

Proof: In proving the assertion we will apply ideas of the Nagano–Sussmann–Jurdjevic theory of attainability.

The first observation is that the vector field

$$f(x) = \{f_u(x) = Ax + Bu\}$$

can be replaced by the vector field

$$\mathcal{F}(x) = \{f_K(x) = Ax + BKx\},$$

if $x \neq 0$. Indeed, for any $u \in \Omega$ one can choose a nonzero component x_i of x and a $K = [k_{l,j}]$ such that $k_{l,j} = 0$ if $j \neq i$ and $k_{l,i} = \frac{u_l}{x_i}$, then $u = Kx$. Actually one has

$$F(x) = \mathcal{F}(x), \quad \text{if } x \neq 0.$$

Moreover, for any $y, z \in \mathbb{R}^n \setminus 0$ there is a trajectory of the original system that does not pass through the origin. This follows from the fact that the origin is normally reachable from any point, see [26], [73]. Then by the surjective mapping theorem, [6], follows that a neighborhood of the origin is reachable by the same switching sequence. Hence, by the linearity of the vector fields, the whole space is reachable with the given switching sequence.

Since the trajectory $x(t)$ does not pass through the origin, the original vector fields $(F(x))$ can be replaced by the new one $(\mathcal{F}(x))$. Moreover, since a given component of $x(t)$ might vanish only a finite times on a finite interval, it follows that the controls K_i of the vector field $F_K(x)$ are piecewise continuous. It follows that every point pair of the manifold $\mathbb{R}^n \setminus 0$ can be joined by a trajectory corresponding to the vector field \mathcal{F} by admissible controls.

It follows that the vector field \mathcal{F} is completely controllable on the manifold $\mathbb{R}^n \setminus 0$. Since complete controllability implies controllability by piecewise constant controls, see [25], [26], it follows that every point pair of the space $\mathbb{R}^n \setminus 0$ can be joined by a trajectory of suitable autonomous switched systems $A_l + B_l K_l$.

Remark 11 *Complete controllability of the vector field \mathcal{F} has a very intuitive geometrical background. Since the solutions of a linear autonomous differential equations realizes some rotations and dilations/compressions in \mathbb{R}^n , it means that for a given point pair (y, z) it is possible to select a finite set of feedbacks such that the resulting set of autonomous systems transform the point y into z for a suitable (finite) switching sequence.*

In order to show that it is possible to select a finite set of autonomous systems that has the (weak) stabilizability property, the compactness argument applied in the proof of Lemma 1 can be repeated.

Indeed, selecting a point y on the unit sphere \mathcal{S} and fixing a point z on the sphere ϵ_1 , there is a trajectory formed by suitable autonomous systems

$A_l + B_l K_l$ that steers y to z , i.e.

$$\Psi(\tau_y, K_y)y = \prod_{j=1}^{L_y} e^{(A_{l_j} + B_{l_j} K_j)\tau_j} y = z.$$

By continuity of $\Psi(\tau_y, K_y)$ for fixed τ_y and K_y there is a neighborhood of y that is mapped in a sufficiently small neighborhood of z , such that $\|\Psi(\tau_y, K_y)\xi\| < \epsilon_2$, with $0 < \epsilon_1 < \epsilon_2 < 1$, see Fig. 3.3.

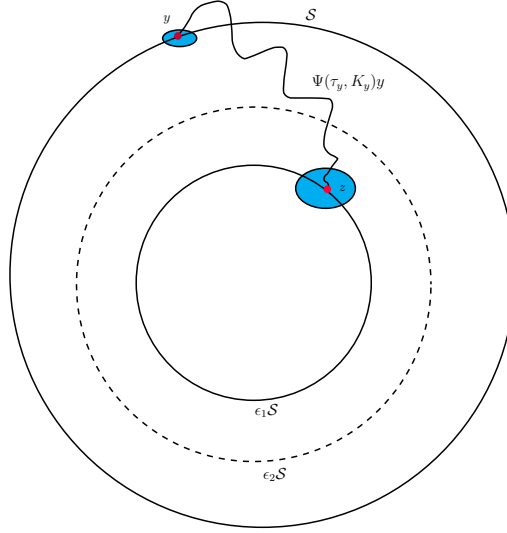


Figure 3.3: Closed-loop stabilizability

These neighborhoods form a covering of the unit sphere, from which it is possible to select a finite one. It follows that it is possible to select a finite set of linear static state feedbacks such, that the resulting set of autonomous system is stabilizable.

The topological argument used in the proof of the Theorem 5 does not give a method to compute the feedbacks. In what follows, by using an (Euler) discretized differential inclusion, a constructive proof will be given for the unconstrained case ($\Omega = \mathbb{R}^m$).

3.3.1 Periodic switching sequence

Lemma 2 *Completely controllable discrete-time linear switched systems with reversible dynamics (nonsingular A_i) can be piecewise linear feedback stabilized using a periodic switching sequence (time dependent switching).*

Proof: This assertion also appears in [86] but its proof is not entirely correct. A corrected proof is as follows: choose a switching sequence σ such that $\text{rank } \mathcal{C}_\sigma = n$. By choosing a nonsingular Schur-stable matrix A_d , one can explicitly construct the piecewise constant inputs that stabilize the time-varying systems obtained by a periodic repetition of the sequence σ with:

$$u^{x_0} = (\mathcal{C}_\sigma^\dagger)(A_d - A_\sigma)x_0, \quad (3.6)$$

where $A_\sigma = A_{s_N} \cdots A_{s_1}$ and M^\dagger denotes a generalized inverse of M . The assertion would follow with feedbacks K_i such that $A_d = \prod_{i=1}^N (A_{s_i} + B_{s_i}K_i)$ provided that the system

$$\tilde{K}_i = K_i \prod_{j=1}^{i-1} (A_{s_j} + B_{s_j}K_j) \quad (3.7)$$

would be solvable for $\tilde{K}_i = P_i(\mathcal{C}_\sigma^\dagger)(A_d - A_\sigma)$ with P_i the projection that gives the i^{th} input from (3.6). This is equivalent with the assertion that the resulting feedback sequence is such that $A_{s_i} + B_{s_i}K_i$ is nonsingular. This is not true in general. However instead of A_d considering its slight perturbation A_c – which is also a Schur matrix – one can obtain the desired feedbacks. This follows from the fact that the determinant is a continuous function of the matrix components. Hence if at a given step $A_{s_i} + B_{s_i}K_i$ would be nonsingular then K_i can be perturbed to $\hat{K}_i = K_i + \epsilon_i K_{i,\epsilon_i}$ such that $A_{s_i} + B_{s_i}\hat{K}_i$ is nonsingular and $\epsilon_i > 0$ is arbitrarily small. Finally one get the matrix $A_c = \prod_{i=1}^N (A_{s_i} + B_{s_i}\hat{K}_i) = A_d + \sum_{i=1}^N \epsilon_i B_{s_i}K_{i,\epsilon} \tilde{A}_i$ with $\tilde{A}_i = \prod_{j=1}^{i-1} (\bar{A}_{s_j} + \bar{B}_{s_j}\hat{K}_j)$, $\tilde{A}_1 = \mathbb{I}$. For sufficiently small ϵ_i the matrix A_c will be also a Schur matrix.

Observe that the number of modes needed for the stabilization is bounded by the length of the switching sequence σ .

Proposition 4 *Completely controllable linear switched systems can be piecewise linear feedback stabilized using a periodic switching sequence.*

Proof: For the proof recall that the reachable set of the Euler discretized differential inclusion approaches uniformly well the reachable set of the original inclusion \mathcal{A}_c , see Proposition 5.3 in [83], i.e. for a given $\epsilon > 0$ and for all ξ in a compact set there is an N_0 independent of ξ such that for each $N > N_0$, $0 \leq j \leq N$ one has $\text{dist}(\mathcal{R}^{jh}(\xi), (I + \frac{T}{N}\mathcal{A}_c)^j(\xi)) < \epsilon$, where dist is the Hausdorff distance. Since for almost all τ one has $\langle A, \mathcal{V} \rangle = \langle \mathbb{I} + \tau A, \mathcal{V} \rangle$, the Euler discretized system will be also completely controllable, moreover there is a common stabilizing sequence σ for the two systems. Applying Lemma 2 for the discretized system one can design feedbacks that ensure arbitrary high decay rates of the closed-loop system. By choosing sufficiently small τ the point $\tilde{A}_\sigma \xi$ will be in a sufficiently small neighborhood of $\bar{A}_\sigma \xi$, where $\tilde{A}_i = A_i + B_i K_i$ and $\bar{A}_i = \mathbb{I} + \tau \tilde{A}_i$ and \bar{A}_σ is a Schur matrix. It follows that the matrix $e^{\tau \tilde{A}_{s_N}} \cdots e^{\tau \tilde{A}_{s_1}}$ will be also a Schur matrix. This proves the assertion.

Remark 12 *Periodical stabilization of switched linear systems was also investigated in [87], where the case when the sum of the individual controllability subspaces gives the whole state space was covered.*

In [86] an LMI condition was given for the synthesis of the stabilizing feedback gains of unconstrained controllable discrete-time linear switching systems. In view of Proposition 4 that result can be directly applied for the stabilization of sampled unconstrained controllable linear switching systems.

This section will be concluded by a slightly extended version of that result by setting LMIs that provide robust stabilization for uncertain systems.

Proposition 5 *Suppose that the uncertain discrete-time switching system $x_{k+1} = A_i(\Delta)x_k + B_i(\Delta)u_k$, $u_k \in \mathbb{R}^m$ is controllable and suppose that there exist a switching sequence $\sigma = (s_1, \dots, s_M)$ such that $\mathcal{R}_\sigma = \mathbb{R}^n$ independently of Δ .*

Then there exist a positive definite matrix S , nonsingular matrices V_i and matrices F_i such that the following LMI is feasible.

$$\begin{bmatrix} S & A_{s_M}V_M + B_{s_M}F_M & \cdots & 0 & 0 \\ (\bullet)^T & V_M + V_M^T & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & V_2 + V_2^T & A_{s_1}V_M + B_{s_1}F_1 \\ 0 & 0 & \cdots & (\bullet)^T & V_1 + V_1^T - S \end{bmatrix} > 0$$

The system can be stabilized with the periodic switching signal defined by σ and the state feedback gains given by $K_i = F_i V_i^{-1}$, $i = 1, \dots, M$.

Proof: The proof of the assertion is the same as in [86] and it is based by a recursive application of the elimination lemma, [15]. Denote by $\sigma_i = (s_i, \dots, s_M)$, then $\bar{A}_{\sigma_i} = \bar{A}_i \bar{A}_{\sigma_{i-1}}$ with $\bar{A}_i = A_{s_i} + B_{s_i} K_i$. By assumption there is an $S > 0$ such that $\bar{A}_{\sigma_1} S \bar{A}_{\sigma_1}^T - S < 0$ which can be written as

$$\begin{bmatrix} \mathbb{I} & -\bar{A}_M(\Delta) \end{bmatrix} \begin{bmatrix} -S & 0 \\ 0 & \bar{A}_{\sigma_{M-1}}(\Delta) S \bar{A}_{\sigma_{M-1}}^T(\Delta) \end{bmatrix} \begin{bmatrix} \mathbb{I} \\ -\bar{A}_M^T(\Delta) \end{bmatrix} < 0 \quad (3.8)$$

By the elimination lemma this inequality is equivalent with:

$$\begin{bmatrix} -S & 0 \\ 0 & \bar{A}_{\sigma_{M-1}}(\Delta) S \bar{A}_{\sigma_{M-1}}^T(\Delta) \end{bmatrix} + \text{Sym}\left\{ \begin{bmatrix} -\bar{A}_M(\Delta) \\ -\mathbb{I} \end{bmatrix} V_M \begin{bmatrix} 0 & \mathbb{I} \end{bmatrix} \right\} \quad (3.9)$$

Repeating the procedure one can obtain the assertion of the proposition.

Remark 13 *Having a polytopic uncertainty, i.e. $A(\Delta) = A_0 + \delta_1 A_1 + \dots + \delta_k A_k$, the LMIs of Proposition 5 form a finite set of condition that can be easily solved.*

3.3.2 Event-driven stabilization

By using the techniques of the previous section it is possible to construct a set of autonomous linear systems that can be asymptotically stabilized, hence it is exponentially stable, see [70]. The problem that remains is that of finding a suitable switching sequence with low complexity that stabilizes the system in "closed-loop", i.e. to solve the event-driven stabilization problem.

The general theory claims that such a feedback (concerning the switching signal) exists. In a fairly general setting a (weak) asymptotic stabilizability result of nonsmooth controlled nonlinear systems can be given in terms of control-Lyapunov functions. By making use of the control-Lyapunov function it is possible to construct a discontinuous time-invariant feedback stabilizer that, when implemented with a sample-and-hold strategy, guarantees semiglobal practical asymptotic stability, [38]. However, these constructions are quite complex.

It turns out that for the stabilizable linear switched systems a stabilizing switching rule (control) can be always given through a conic partition $\mathbb{R}^n = \bigcup_{k=1}^N \mathcal{C}_k$ of the state space where on each cone \mathcal{C}_k the mode $i_k \in S$ is active.

Based on the ideas presented in [52] it was proved that the (weak) asymptotic stabilizability of some switched autonomous linear systems by means of an event driven switching strategy can be formulated in terms of a conic partition of the state space, see [47],[49]. The construction, however, is based on the existence of a convex (nonsmooth) Lyapunov function while [11] provides examples for stable switched systems that does not have a convex Lyapunov function. Nevertheless, the example from [11] can be also stabilized by a conic partition based switching.

As a solution based on a single Lyapunov function usually is hard to be found efforts have been devoted to the development of multiple Lyapunov functions, e.g. [17, 46]. For numerical and practical reasons, the individual functions are usually chosen to be quadratic, see e.g. [34] and for a recent survey on composite quadratic Lyapunov techniques [30]. The conditions of stabilization are derived as bilinear matrix inequalities (BMI), i.e. in terms of nonconvex conditions.

The general result regarding the stabilizability in terms of a conic partition based switching rule comes from optimization theory. For an exponentially stable linear autonomous switched system the quadratic optimization problem

$$\min_{\sigma} \int_0^{\infty} x^T Q_{\sigma(t)} x dt \quad (3.10)$$

with positive definite matrices $Q_i, i \in S$ has a solution. Moreover, the solution is given by a conic partition based switching law, see [21, 58]. The construction of such a partition is a numerically intensive process.

3.3.3 The general stabilizability problem

For convex processes stabilizability can be given in terms of the spectral properties of the process, see [61]. Unfortunately the differential inclusion associated to general switched linear systems is not a convex process and a general characterization of stabilizability is still lacking.

Even questions that are trivial for LTI systems are hard to be decided for switched systems. As an example: consider the Kalman decomposition

of the linear switched system defined by the invariant (controllability) subspace $\mathcal{R}_{\mathcal{A},\mathcal{B}}$. As opposed to the LTI case in general there are no feedbacks that render such a decomposition diagonal. This fact does not affect the controllability on $\mathcal{R}_{\mathcal{A},\mathcal{B}}$, however it makes a stability analysis less transparent. Moreover, even if there would be a diagonal decoupling between the controllable and uncontrollable part, stabilizability implies that both parts should be stabilizable by using the same switching sequence, which is hard to decide.

3.4 Controllability analysis of bimodal systems

Note, that in Theorem 3 the subspace $\mathcal{R}_{\mathcal{A},\mathcal{B}}$ is the minimal subspace invariant for all of the A_i s containing $\mathcal{B} = \sum_{i=0}^s \text{Im} B_i$.

The bimodal system can be transformed, via a state transform and suitable feedbacks, to

$$\dot{\eta}_1 = \begin{cases} P_{1,1}\eta_1 + \tilde{R}_1 y_s + \tilde{Q}_1 u_1 & \text{if } y_s \geq 0 \\ P_{2,1}\eta_1 + \tilde{R}_2 y_s + \tilde{Q}_2 u_2 & \text{if } y_s \leq 0 \end{cases}, \quad (3.11)$$

$$\dot{\eta}_2 = \begin{cases} P_{1,2}\eta_2 + R_1 y_s & \text{if } y_s \geq 0 \\ P_{2,2}\eta_2 + R_2 y_s & \text{if } y_s \leq 0 \end{cases}, \quad (3.12)$$

$$\dot{y}_s = v, \quad (3.13)$$

where, by Theorem 3, subsystem (3.11) is controllable on \mathcal{C} using open-loop switchings. Thus, this decomposition can be viewed as a controllability decomposition of the bimodal LTI system where the study of the controllability of the original bimodal system reduces to controllability of the bimodal system formed by (3.12) and (3.13).

Remark 14 When $y_s = 0$, i.e., on \mathcal{C} , subsystems (3.11) does not contain y_s and the switching law must be defined externally. However for linear switching systems there exist a universal switching sequence that provides complete controllability, hence the switching sequence is fixed and a fundamental solution of (3.11) (as a linear time varying system) is well defined. Therefore, by linearity, the controllability of (3.11) is not affected by the values of y_s .

Lemma 3 The bimodal system (2.15), (2.16) is completely controllable if and only if the subsystem defined by (3.12), (3.13) is completely controllable.

Proof The necessity is obvious. For the sufficiency it is enough to consider the reachability case, i.e., the situation when $x_0 = 0$ and a given x_f is to be reached. Decompose x_f into $\eta_{1,f}$ and $(\eta_{2,f}, y_{s,f})$ according to (3.11) and (3.12),(3.13). Since the OLLS (3.11) is completely controllable, there is a finite switching sequence, see [70], and suitable inputs u_1, u_2 that steers the origin according to (3.11) to $\eta_{1,f}$. Let us denote these inputs by u_η . The switching sequence can be realized by a suitable y_s^η that has sign changes at the required time instances, e.g., a modulated sine signal. Let us denote by v_η one of the controls that realizes y_s^η . By linearity and complete controllability of (3.12),(3.13) there are points $(\eta_{2,o}, y_{s,o})$ from which the system (3.12),(3.13) is steered into $(\eta_{2,f}, y_{s,f})$ applying v_η . Let us denote by v_o the input that steers (3.12),(3.13) from the origin to $(\eta_{2,o}, y_{s,o})$. During this the inputs u_i are maintained at zero, i.e., $\eta_1 = 0$. It follows that applying the inputs $(0, u_\eta), (v_o, v_\eta)$ one can steer the origin into $\eta_{1,f}$ and $(\eta_{2,f}, y_{s,f})$, i.e., into x_f .

Having the decomposition (3.12),(3.13) for a bimodal system it is immediate that if the system is controllable then the input constrained open-loop switching system of the type

$$\dot{\eta} = P_i \eta + \bar{R}_i w, \quad i \in \{1, 2\}, \quad w \geq 0 \quad (3.14)$$

with $\bar{R}_i = (-1)^{i+1} R_i$ is also controllable. Consulting the result of [28], i.e., the case $P_1 = P_2$, it is apparent that the controllability condition of the bimodal system is equivalent to the input constrained controllability condition of the corresponding open-loop system given by (3.14). It is less apparent, but this consequence also holds for the case presented in [79].

A separation theorem

The bimodal system (3.12), (3.13) can be seen as a dynamic extension of

$$\dot{\eta}_2 = P_{i,2} \eta_2 + \bar{R}_{i,2} w, \quad i \in \{1, 2\}, \quad w \geq 0 \quad (3.15)$$

see [33]. Controllability of the dynamically extended system, provided that the original system was controllable, is by far non-trivial issue though for smooth vector fields it was proved in [75, 67]. For linear systems it is straightforward for unconstrained input case. This can be verified by checking the Kalman rank condition of the extended system, however this result cannot be directly applied here, since the input is signed constrained.

Lemma 4 *If the points η_0 and η_f can be connected by a trajectory of the linear system $\dot{\eta} = P\eta + R w$ using nonnegative control $w \geq 0$ then, for a given r , they can be also connected using a smooth nonnegative control $\omega \geq 0$ with prescribed end points, i.e., $\omega^{(k)}(0) = \omega_{0,k}$ and $\omega^{(k)}(T_f) = \omega_{T_f,k}$ for $k = 0, 1, \dots, r$.*

Proof The proof of the assertion is an adaptation of the proof for controllability by smooth controls given in Chapter 5., Theorem 4 of [36]. The main points of the proof are the following: for a linear system every accessible point is normally accessible (i.e., by using piecewise constant controls). Consider the control formed by the sequence $(\mathbf{w}, \hat{\mathbf{t}}) = \{(w_1, \hat{t}_1), \dots, (w_F, \hat{t}_F)\}$ where the control $w_i \geq 0$ is applied for a duration of \hat{t}_i , that steers η_0 to η_f . By the inverse mapping theorem, there exist functions t_i defined on a neighborhood V of η_f such that the sequence $\{(w_1, t_1(z)), \dots, (w_F, t_F(z))\}$ steers η_0 to z for all $z \in V$. Denote by $\tau_i = \sum_{l=1}^i t_l$ and for any sufficiently small $\epsilon > 0$ consider the smooth nonnegative control $\omega(t, z, \epsilon)$ defined by

$$\omega(t, z, \epsilon) = \begin{cases} \beta_1(t, z) & \text{if } t \in [0, \epsilon] \\ w_1 & \text{if } t \in [\epsilon, \tau_1(z) - \epsilon] \\ w^* & \text{if } t \in [\tau_1(z) - \epsilon, \tau_1(z) + \epsilon] \\ \vdots & \\ w_F & \text{if } t \in [\tau_{F-1}(z) + \epsilon, \tau_F(z) - \epsilon] \\ \beta_F(t, z) & \text{if } t \in [\tau_F(z) - \epsilon, \tau_F(z)], \end{cases} \quad (3.16)$$

where $w^* = (1 - \alpha_1(t, z))w_1 + \alpha_1(t, z)w_2$ and α_i, β_j are smooth, nonnegative, increasing functions in t for each z in the interval $[\tau_i^-, \tau_i^+] := [\tau_i(z) - \epsilon, \tau_i(z) + \epsilon]$, with end conditions $\alpha_i(\tau, z) = 0$, $\partial_{t^k}^k \alpha_i(\tau, z) = 0$ at $\tau \in \{\tau_i^-, \tau_i^+\}$ and $k \geq 1$. The same end conditions are imposed for β_1 at t_1^+ and for β_F at t_F^- . We also impose $\partial_{t^k}^k \beta_1(0, z) = \omega_{0,k}$ and $\partial_{t^k}^k \beta_F(\tau_F(z), z) = \omega_{T_f,k}$ for $k = 0, 1, \dots, r$. If we denote the associated integral curve by $\Phi_\epsilon(t)$ then one has that $\lim_{\epsilon \rightarrow 0} \Phi_\epsilon(\tau_F(z)) = z$ in some neighborhood of η_f . The assertion of the Lemma follows by a fix point argument, for details see [36] or [75].

We can formulate one of the main results of this paper using this Lemma:

Proposition 6 *The bimodal system given by (3.12) and (3.13) is controllable if and only if the input constrained open-loop switching system (3.15) is controllable.*

3.5 Stabilizability of bimodal systems

The bimodal system (2.8) is said to be stabilizable if any initial state can be asymptotically steered to the origin by a suitable admissible input u , i.e., for all $x_0 \in \mathbb{R}^n$ there exist a solution $x(t)$ of the bimodal system such that $\lim_{t \rightarrow \infty} x(t) = 0$.

Let us first examine bimodal systems with continuous dynamics. In view of Proposition 6 these systems are equivalent with an LTI system with two sign constrained inputs. Starting from this observation one has the following result:

Proposition 7 *If the bimodal system has continuous dynamics, i.e., $P_1 = P_2 = P$, then the bimodal system (3.12), (3.13) is stabilizable if and only if the corresponding sign constrained open-loop switching system (3.1) is stabilizable.*

Proof: The necessity is obvious. For sufficiency let us recall the following basic fact: for a stabilizable LTI system, in particular for the sign constrained system $\dot{x} = Px + [\bar{R}_1 \bar{R}_2]w$, $w \geq 0$, there exist numbers $\alpha > 0$ and $\gamma > 0$ such that for any point x_0 a trajectory of system satisfying the condition $x(0) = x_0$ and

$$\|x(t)\| \leq \alpha \|x_0\| e^{-\gamma t} \quad t \geq 0 \quad (3.17)$$

can be found, see [60]. It follows, that for this trajectory one has $\int_0^\infty w < \infty$, i.e., $\lim_{t \rightarrow \infty} w(t) = 0$. Moreover this w can be chosen to be continuous, see [60], i.e., the implied switching sequence induced by the sign changes of w is piecewise constant. Then the construction leading to Proposition 6 can be applied and it follows, that the corresponding bimodal system, which has the state $\begin{bmatrix} x \\ w \end{bmatrix}$, is also stable.

In [29] one can find the following characterization of the stabilizability of a sign constrained LTI system:

Theorem 6 *The system*

$$\dot{x} = Px + Rw \quad w \in \mathbb{R}_+^2 \quad (3.18)$$

is stabilizable if and only if the unconstrained system is stabilizable and all real eigenvectors v of P^T corresponding to a nonnegative eigenvalue of P^T have the property that $R^T v$ has both positive and negative components.

Remark 15 *Since for an unconstrained LTI system controllability and stabilizability are equivalent, the first conditions of the theorem means, that the system has to satisfy the Kalman rank condition.*

An equivalent result was given in [60], where a method for the construction of the stabilizing feedback was also presented.

If the more severe conditions of small time local controllability are satisfied, then Lipschitz continuous piecewise linear stabilizing feedback can be constructed, see [80].

The conditions of the Theorem (6) are satisfied for controllable systems, see Theorem 4.

The general case is more difficult and is beyond the scope of this paper to enter in more details regarding the solvability of the problem. We conclude this section with a result that provides a sufficient condition for stabilizability:

Proposition 8 *If the bimodal system (3.12), (3.13) is globally controllable, then it is asymptotically stabilizable.*

Proof: By controllability and by Theorem 4 one has that from any initial state x_0 there is a control that steers the point to the origin in a finite time, say T . By the finite switching property, Proposition 1, at time T a well defined system is active. Setting the input $u = 0$ for $T > 0$ the system is maintained in the origin, i.e. the system is stable.

Chapter 4

Conclusions

- This report investigates controllability of linear time invariant (LTI) switching systems using nonnegative inputs. The paper gives algebraic conditions that guarantees global controllability for this class of systems. It is shown that if the system is globally controllable then the number of necessary switchings to control the system is bounded. A fairly generalized and modified view of the previous results reported in literature concerning controllability of switching systems was given.
- The report considers the problem of event driven stabilization of linear controlled switched systems and claims that if the system is completely controllable then it is stabilizable by generalized piecewise linear feedback, i.e. it is stabilizable by applying event driven switchings for a finite set of linear autonomous systems obtained by applying a suitable set of linear feedback.

The obtained results are general enough to include the class of linear controlled switched systems with sign constrained control inputs.

For the class of completely controllable unconstrained linear switching systems one has not only closed-loop stabilizability but also closed-loop exponential stabilizability. It is a subject of further research to explore what type of performance bounds exists concerning the convergence rate.

- The report considers the problem of finding a constructive algorithm for (closed-loop) stabilizability of controlled linear switched systems with unconstrained inputs. It was shown that the completely controllable

sampled switching system can be robustly stabilized (against disturbances and model uncertainties) with suitable linear feedbacks and a periodic switching strategy. The suitable feedback gains are computed via a set of LMIs. The algorithm relays on the computation of a suitable σ such that $\mathcal{R}_\sigma = \mathbb{R}^n$, which can be relatively easily constructed. However, obtaining a sequence with minimal length is an open problem yet.

- This report also investigates controllability of a class of bimodal linear time invariant (LTI) systems pointing to the relevant structures of the problem. It was shown that for a certain class controllability is equivalent with controllability of an open-loop switching system using nonnegative controls, i.e. to the controllability of a constrained open-loop switching system. The problem of stabilizability it was also addressed.

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List of Symbols, Abbreviations and Acronyms

LTI linear time invariant (system)

LTV linear time varying (system)

GAC globally asymptotically controllable (system)

GPLFS generalized piecewise linear feedback stabilizability

LMI linear matrix inequality

BMI bilinear matrix inequality

\mathbb{R} set of real numbers

\mathbb{R}_+ set of nonnegative real numbers

\mathbb{Z} set of integers

\mathbb{N} set of nonnegative integers